14 Perpendicularity and Angle Congruence

Definition (acute angle, right angle, obtuse angle, supplementary angles, complementary angles)
An acute angle is an angle whose measure is less than 90. A right angle is an angle whose measure is 90. An obtuse angle is an angle whose measure is greater than 90. Two angles are supplementary if the sum of their measures is 180. Two angles are complementary if the sum of their measures is 90.

Definition (linear pair of angles, vertical pair of angles)
Two angles $\measuredangle ABC$ and $\measuredangle CBD$ form a linear pair if $A-B-D$. Two angles $\measuredangle ABC$ and $\measuredangle A'BC'$ form a vertical pair if their union is a pair of intersecting lines. (Alternatively, $\measuredangle ABC$ and $\measuredangle A'BC'$ form a vertical pair if either $A-B-A'$ and $C-B-C'$, or $A-B-C'$ and $C-B-A'$.)

Theorem If C and D are points of a protractor geometry and are on the same side of $\overleftrightarrow{AB}$ and $m(\measuredangle ABC) < m(\measuredangle ABD)$, then $C \in \text{int}(\measuredangle ABD)$.

1. Prove the above theorem.

[Theorem 5.3.1, page 104]

Theorem (Linear Pair Theorem). If $\measuredangle ABC$ and $\measuredangle CBD$ form a linear pair in a protractor geometry then they are supplementary.

2. Prove the above theorem.

[Theorem 5.3.2, page 105]

3. If $A'-V-A$, $B'-V-B$, and $\measuredangle AVB$ is a right angle, then each of $\measuredangle AVB'$, $\measuredangle A'VB$, and $\measuredangle A'VB'$ is a right angle.

Theorem In a protractor geometry, if $m(\measuredangle ABC) + m(\measuredangle CBD) = m(\measuredangle ABD)$, then $C \in \text{int}(\measuredangle ABD)$.

4. Prove the above theorem.

[Theorem 5.3.3, page 106]

Note that the result about distance that corresponds to above Theorem is false. If $AB + BC = AC$ it need not be true that $B \in \text{int}(AB)$.

Theorem In a protractor geometry, if A and D lie on opposite sides of $\overleftrightarrow{BC}$ and if $m(\measuredangle ABC) + m(\measuredangle CBD) = 180$, then $A-B-D$ and the angles form a linear pair.

5. Prove the above theorem.

Definition (perpendicular lines, perpendicular rays, perpendicular segments)
Two lines ℓ and ℓ' are perpendicular (written $\ell \perp \ell'$) if $\ell \cup \ell'$ contains a right angle. Two rays or segments are perpendicular if the lines they determine are perpendicular.

6. If a is a segment, ray, or line and b is a segment, ray, or line, then $a \perp b$ implies $b \perp a$.

Theorem If P is a point on line ℓ in a protractor geometry, then there exists a unique line through P that is perpendicular to ℓ.

7. Prove the above theorem.

8. In the Poincaré Plane, find the line through $B(3,4)$ that is perpendicular to the line ${}_0L_5 = \{(x,y) \in \mathbb{H} \mid x^2 + y^2 = 25\}$.

[Example 5.3.6, page 107]

Corollary In a protractor geometry, every line segment $\overline{AB}$ has a unique perpendicular bisector; that is, a line $\ell \perp \overline{AB}$ with $\ell \cap \overline{AB} = \{M\}$ where M is the midpoint of $\overline{AB}$.

9. Prove the above corollary.

Theorem In a protractor geometry, every angle $\measuredangle ABC$ has a unique angle bisector that is, a ray $\overrightarrow{BD}$ with $D \in \text{int}(\measuredangle ABC)$ and $m(\measuredangle ABD) = m(\measuredangle DBC)$.

10. Prove the above theorem.

Definition (angle congruence)
In a protractor geometry $\{\mathcal{S}, \mathcal{L}, d, m\}$, $\measuredangle ABC$ is congruent to $\measuredangle DEF$ (written as $\measuredangle ABC \cong \measuredangle DEF$ if $m(\measuredangle ABC) = m(\measuredangle DEF)$.

11. Congruence of angles is an equivalence relation on the set of all angles.

12. Prove that any two right angles in a protractor geometry are congruent.

Theorem (Vertical Angle Theorem). In a protractor geometry, if $\measuredangle ABC$ and $\measuredangle A'BC'$ form a vertical pair then $\measuredangle ABC \cong \measuredangle A'BC'$.

13. Prove the above theorem.

Theorem (Angle Construction Theorem). In a protractor geometry, given $\measuredangle ABC$ and a ray $\overrightarrow{ED}$ which lies in the edge of a half plane H_1, then there exists a unique ray $\overrightarrow{EF}$ with $F \in H_1$ and $\measuredangle ABC \cong \measuredangle DEF$.

14. Prove the above theorem.

Theorem (Angle Addition Theorem). In a protractor geometry, if $D \in \text{int}(\measuredangle ABC)$, $S \in \text{int}(\measuredangle PQR)$, $\measuredangle ABD \cong \measuredangle PQS$, and $\measuredangle DBC \cong \measuredangle SQR$, then $\measuredangle ABC \cong \measuredangle PQR$.

15. Prove the above theorem.

Theorem (Angle Subtraction Theorem). In a protractor geometry, if $D \in \text{int}(\measuredangle ABC)$, $S \in \text{int}(\measuredangle PQR)$, $\measuredangle ABD \cong \measuredangle PQS$, and $\measuredangle ABC \cong \measuredangle PQR$, then $\measuredangle DBC \cong \measuredangle SQR$.

16. Prove the above theorem.

17. Show that if $\triangle ABC$ is in Poincaré plane with $A(0,1)$, $B(0,5)$, and $C(3,4)$ (this triangle we had earlier), then $(AC)^2 \neq (AB)^2 + (BC)$. Thus the Pythagorean Theorem need not be true in a protractor geometry.

18. In $\mathcal{H}$ find the angle bisector of $\measuredangle ABC$ if $A = (0,5)$, $B = (0,3)$ and $C = (2,\sqrt{21})$.

19. Prove that in a protractor geometry $\measuredangle ABC$ is a right angle if and only if there exists a point D with $D-B-C$ and $\measuredangle ABC \cong \measuredangle ABD$.

20. In the Taxicab Plane let $A = (0,2)$, $B = (0,0)$, $C = (2,0)$, $Q = (-2,1)$, $R = (-1,0)$ and $S = (0,1)$. Show that $\overline{AB} \cong \overline{QR}$, $\measuredangle ABC \cong \measuredangle QRS$, and $\overline{BC} \cong \overline{RS}$. Is $\overline{AC} \cong \overline{QS}$?

15 Euclidean and Poincaré Angle Measure

In this optional section we shall carefully verify that the Euclidean and Poincaré angle measures defined in Section 13 actually satisfy the axioms of an angle measure. The key step will be the construction of an inverse cosine function. This will involve techniques quite different from those of the rest of this course. As a result, you may choose to omit this section knowing that the only results that we will use in the sequel are that m_E and m_T are angle measures and that the cosine function is injective. On the other hand, it is interesting to see a variety of mathematical techniques tied together to develop one concept as is done in this section. The material on the construction of Euclidean angle measure is taken from Parker [1980].

Precisely what are we assuming in this section? We are assuming the standard facts about differentiation and integration but nothing about trigonometric functions. This will force us to consider the notion of an improper integral in order to define the inverse cosine function. Since general results about differential equations may not be familiar to the reader, we shall need to develop some very specific theorems regarding the solutions of $y'' = -y$. (In calculus we learned that both $sin(x)$ and $cos(x)$ are solutions of this differential equation. That is why we are interested in this equation.)

Definition (improper integral)
Let $f(t)$ be a function which is continuous for $c \le t < d$ and which may not be defined at $t = d$. Then the improper integral $\int_c^d f(t)\,\mathrm{d}t$ converges if $\lim_{b\to d^-} f(t)\,\mathrm{d}t$ exists. In this case, we say $\lim_{b\to d^-} f(t)\,\mathrm{d}t = \int_c^d f(t)\,\mathrm{d}t$.

Lemma The improper integral $\displaystyle\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ converges. [Lemma 5.4.1, page 110]

A similar argument shows that the improper integral $\int_{-1}^0 \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ converges so that $\int_0^{-1} \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ also exists. We define a number p to be twice the value of the integral in above lemma: ...

...(see book, pages 109-123)...